Concluding remarks

In § I, we explained a geometric method for constructing crystal families of space E^n out of the different families of spaces E^1 and E^2 and the gZ-irr. families. The irreducible families were studied in detail in § II and five types of irreducibility were exhibited.

This method enables us to give a name to the crystal family connected to their construction. Then, we easily deduce the WPV symbols of the holohedries. As for the gZ-irr. family, the name explains their construction or sometimes the PSOs that characterize the families.

In forthcoming papers, we state systematic rules for giving a name to crystal families and we list all crystal families of spaces E^1, E^2, E^3, E^4, E^5 (paper XII; Weigel & Veysseyre, 1993), space E^6 (paper XIII) and space E^7 (paper XIV).

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Crystallography, Geometry and Physics in Higher Dimensions. XII. Counting and General Nomenclature of N-Dimensional Crystal Families: Application from One- to Five-Dimensional Spaces

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Abstract

In paper XI [Veysseyre, Weigel & Phan (1993). *Acta Cryst.* A**49**, 481–486], the definition was given of the geometrically Z-irreducible and the geometrically Z-reducible crystal families of the n-dimensional space and a general method was described for constructing all crystal families. In this paper, systematic rules are stated for giving names to the crystal families and these are listed for spaces E^1, E^2, E^3, E^4 and E^5.

Introduction

The aim of this paper is to explain how the geometrically Z-reducible (gZ-red.) property of a crystal family enables us to give a name to these families through some examples. This paper is divided into three sections:

(i) § 1 is concerned with the counting (*i.e.* the number) of all crystal families from one- to seven-dimensional spaces;

(ii) § 2 expresses strict rules that lead us to assign correct names to the gZ-red. crystal families of E^n and mainly to spaces of dimensions one to five;

(iii) § 3 explains the connection between the transitive crystallographic rotations (Bertaut, 1988) and the geometrically Z-irreducible (gZ-irr.) families and explains the choice of their names too (for the one- to five-dimensional spaces).

I. Counting of all crystal families of space E^n

The study of all partitions of space E^n into subspaces that are two-by-two orthogonal enables us to describe all gZ-red. crystal families. The type of the gZ-reducibility is given by the dimension of each space occurring in the splitting of space E^n and by the type of the irreducibility of the crystal family (Veysseyre, Weigel & Phan, 1993).

For instance, if we consider the partition $E^6 = E^3 \oplus E^2 \oplus E^1$, we say that the type of crystal family built in this way is $(3)+(2)+1$, where (3) means 3 or $\overline{1,1,1}$ and (2) means 2 or $\overline{1,1}$. A general formula will be given § 2. In fact, we can select in space E^3 one of the two gZ-irr. families and in space E^2 one of the three gZ-irr. families (Table 1) and obviously in space E^1 the only gZ-irr. family, *viz* the segment. So, we

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Table 1. *Names of the crystallographic families and their geometrical Z-reducibility or irreducibility*

(i) Number of the crystal family [nomenclature of Brown *et al.* (1978) for $n=4$ and of Plesken (1981) for $n=5$]; (ii) name of the family; (iii) symbol of the holohedry; (iv) order of the holohedry; (v) type of the geometrical Z reducibility or irreducibility of the family.

Two dimensions

(i)	(ii)	(iii)	(iv)	(v)
I	Oblic (oblique)	2	2	$\overline{1,1}$
II	Rectangle (rectangular)	$m\perp m$ or $(2mm)$	4	$1+1$
III	Square (tetragon)	$4mm$	8	2
IV	Hexagon	$6mm$	12	2

Three dimensions

(i)	(ii)	(iii)	(iv)	(v)
I	Triclinic	$\bar{1}$	2	$\overline{1,1,1}$
II	Oblical (monoclinic)	$2\perp m$ or $(2/m)$	4	$\overline{1,1}+1$
III	Orthorhombic	$m\perp m\perp m$ or $(2/mmm)$	8	$1+1+1$
IV	Tetragonal	$4mm\perp m$ or $(4/mmm)$	16	$2+1$
V	Hexagonal	$6mm\perp m$ or $(6/mmm)$	24	$2+1$
VI	Cubic	$m\bar{3}m$	48	3

Table 2. *Names of the crystallographic families and their geometrical Z-reducibility or irreducibility*

(i) Number of the crystal family [nomenclature of Brown *et al.* (1978) for $n=4$ and of Plesken (1981) for $n=5$]; (ii) name of the family; (iii) symbol of the holohedry; (iv) order of the holohedry; (v) type of the geometrical Z-reducibility or irreducibility of the family.

Four dimensions

(i)	(ii)	(iii)	(iv)	(v)
I	Hexaclinic	$\bar{1}_4$	2	$\overline{1,1,1,1}$
II	Triclinic-al	$\bar{1}\perp m$	4	$\overline{1,1,1}+1$
III	Di oblic	$2\perp 2$	4	$\overline{1,1}+\overline{1,1}$
IV	Oblic rectangle	$2\perp m\perp m$	8	$\overline{1,1}+1+1$
V	Orthotopic 4	$m\perp m\perp m\perp m$	16	$1+1+1+1$
VI	Square oblic	$4mm\perp 2$	16	$2+\overline{1,1}$
VII	Hexagon oblic	$6mm\perp 2$	24	$2+\overline{1,1}$
VIII	Diclinic di square	44^*	4	$\overline{2,2'}$
IX	Diclinic di hexagon	66^*	6	$\overline{2,2'}$
X	Square rectangle	$4mm\perp m\perp m$	32	$2+1+1$
XI	Hexagon rectangle	$6mm\perp m\perp m$	48	$2+1+1$
XII	Monoclinic di square	$2, 44^*, 2$	8	$\overline{2,2}$
XIII	Monoclinic di hexagon	$2, 66^*, 2$	12	$\overline{2,2}$
XIV	Di square	$4mm\perp 4mm$	64	$2+2$
XV	Hexagon square	$6mm\perp 4mm$	96	$2+2$
XVI	Di hexagon	$6mm\perp 6mm$	144	$2+2$
XVII	Cubic-al	$m\bar{3}m\perp m$	96	$3+1$
XVIII	Monoclinic di iso square (octodic)	$88, 2$	16	$4'$
XIX	Decadic	$10_2, 2$	20	$4'$
XX	Monoclinic di iso hexagon (dodecadic)	$12_2, 2$	24	$4'$
XXI	Di iso hexagon	$6mm\perp 6mm, 12_2$	288	4
XXII	Rhombotopic $-\frac{1}{4}$	$\bar{4}3m, 10_2$	240	4
XXIII	Hypercubic 4	$m\bar{3}m, 88$	384	4

Five dimensions

(i)	(ii)	(iii)	(iv)	(v)
I	Decaclinic	$\bar{\bar{1}}(\bar{1}_5)$	2	$\overline{1,1,1,1,1}$
II	Hexaclinic-al	$\bar{1}_4\perp m$	4	$\overline{1,1,1,1}+1$
III	Triclinic oblic	$\bar{1}\perp 2$	4	$\overline{1,1,1}+\overline{1,1}$
IV	Triclinic rectangle	$\bar{1}\perp m\perp m$	8	$\overline{1,1,1}+1+1$
V	Di oblic-al	$2\perp 2\perp m$	8	$\overline{1,1}+\overline{1,1}+1$
VI	Triclinic square	$\bar{1}\perp 4mm$	16	$\overline{1,1,1}+2$
VII	Triclinic hexagon	$\bar{1}\perp 6mm$	24	$\overline{1,1,1}+2$
VIII	Oblic orthorhombic	$2\perp m\perp m\perp m$	16	$\overline{1,1}+1+1+1$
IX	Orthotopic 5	$m\perp m\perp m\perp m\perp m$	32	$1+1+1+1+1$
X	Square oblic-al	$4mm\perp 2\perp m$	32	$2+\overline{1,1}+1$
XI	Hexagon oblic-al	$6mm\perp 2\perp m$	48	$2+\overline{1,1}+1$
XII	Diclinic di square-al	$44^*\perp m$	8	$\overline{2,2'}+1$
XIII	Diclinic di hexagon-al	$66^*\perp m$	12	$\overline{2,2'}+1$
XIV	Square orthorhombic	$4mm\perp m\perp m\perp m$	64	$2+1+1+1$
XV	Hexagon orthorhombic	$6mm\perp m\perp m\perp m$	96	$2+1+1+1$
XVI	Monoclinic di square-al	$2, 44^*, 2\perp m$	16	$\overline{2,2}+1$
XVII	Monoclinic di hexagon-al	$2, 66^*, 2\perp m$	24	$\overline{2,2}+1$
XVIII	Cubic oblic	$m\bar{3}m\perp 2$	96	$3+\overline{1,1}$
XIX	Di square-al	$4mm\perp 4mm\perp m$	128	$2+2+1$
XX	Hexagon square-al	$6mm\perp 4mm\perp m$	192	$2+2+1$
XXI	Di hexagon-al	$6mm\perp 6mm\perp m$	288	$2+2+1$
XXII	Cubic rectangle	$m\bar{3}m\perp m\perp m$	192	$3+1+1$
XXIII	Monoclinic di iso square-al (octodic-al)	$88, 2\perp \mathrm{m}$	32	$4'+1$
XXIV	Decadic-al	$10_2, 2\perp m$	40	$4'+1$
XXV	Monoclinic di iso hexagon-al (dodecadic-al)	$12_2, 2\perp m$	48	$4'+1$
XXVI	Cubic square	$m\bar{3}m\perp 4mm$	384	$3+2$
XXVII	Cubic hexagon	$m\bar{3}m\perp 6mm$	576	$3+2$
XXVIII	Di iso hexagon-al	$6mm\perp 6mm, 12_2\perp m$	576	$4+1$
XXIX	(Rhombotopic $-\frac{1}{4}$)-al	$\bar{4}3m, 10_2\perp m$	480	$4+1$
XXX	Hypercubic 4-al	$m\bar{3}m, 88\perp m$	768	$4+1$
XXXI	Rhombotopic $-\frac{1}{5}$	$(\bar{4}3m, 10_2)\overline{\overline{36}}$	1440	5
XXXII	Hypercubic 5	$(m\bar{3}m, 8_2)\overline{\overline{55}}$	3840	5

obtain $2\times 3=6$ gZ-red. families of space E^6, which are: the triclinic oblic-al family of type $\overline{1,1,1}+\overline{1,1}+1$; the triclinic square-al and the triclinic hexagonal families of type $\overline{1,1,1}+2+1$; the cubic oblic-al family of type $3+\overline{1,1}+1$; the cubic square-al and the cubic hexagonal families of type $3+2+1$. All partitions of space E^6 are to be studied similarly.

We then add the gZ-irr. families for each space E^n. This number depends on the number n; it is different if n is an odd or even number and it depends on the possible crystal point-symmetry operations (PSOs) of this space, *i.e.* of Euler indicatrix $\varphi(n)$ (§ 3).

The number of each crystal family, its name, the Weigel–Phan–Veysseyre (WPV) symbols of its

Table 3. *Number of crystal families according to the types of decomposition*

For each dimension of space, we give the different types of decomposition and on the same line the number of crystal families belonging to this type. The last line gives the number of gZ-irr. crystal families, *e.g.* there are 13 gZ-irr. crystal families in space E^6 and 3 in space E^7. For each type of splitting of space, we explain how we find the number of crystal families: either a product of two numbers if the space dimensions are different or the number of combinations with repetitions if the space dimensions are the same.

E^1		E^2		E^3		E^4		E^5		E^6		E^7	
E^1	1	$E^1\oplus[E^1]$	1	$E^1\oplus[E^2]$	4	$E^1\oplus[E^3]$	6	$E^1\oplus[E^4]$	23	$E^1\oplus[E^5]$	32	$E^1\oplus[E^6]$	91
	—	E^2	3	E^3	2	$E^2\oplus E^2$	6	$E^2\oplus E^3$	6	$E^2\oplus E^4$	33	$E^2\oplus E^5$	9
	1		—		—		11	E^5	3	$E^3\oplus E^3$	3	$E^3\oplus E^4$	22
			4		6		—		—	$E^2\oplus E^2\oplus E^2$	10	$E^2\oplus E^2\oplus E^3$	12
							23		32	E^6	13	E^7	3
											—		—
											91		137
						$6=\binom{3+2-1}{2}$		$6=3\times2$		$33=3\times11$		$9=3\times3$	
										$3=\binom{2+2-1}{2}$		$22=2\times11$	
										$10=\binom{3+3-1}{3}$		$12=\binom{3+2-1}{2}\times2$	

holohedries as well as its order and the type of reducibility or irreducibility are listed in Table 1 for $n=2$ and $n=3$ and Table 2 for $n=4$ and $n=5$.

Table 3 gives the number of crystal families according to their types of splitting. For example, in the first line is the entry $[E^3]\oplus E^1$ (space E^4). The symbol $[E^3]$ denotes all the crystal families of space E^3, *i.e.* six crystal families. Therefore, $[E^3]\oplus E^1$ includes three types of splitting:

(i) $E^3\oplus E^1$: type $3+1$, which includes two crystal families; triclinic-al and cubic-al;

(ii), (iii) $[E^2]\oplus E^1\oplus E^1$, which is the abridged writing of $E^2\oplus E^1\oplus E^1$ and $E^1\oplus E^1\oplus E^1\oplus E^1$.

The splitting $E^2\oplus E^1\oplus E^1$ gives three crystal families of type $2+1+1$: oblic rectangle, square rectangle and hexagon rectangle (Table 2). The splitting $E^1\oplus E^1\oplus E^1\oplus E^1$ gives one crystal family of type $1+1+1+1$, which is the orthotopic 4. Therefore, there are $2+3+1=6$ crystal families, which is the number shown in Table 3 for $[E^3]\oplus E^1$.

II. Names of the gZ-red. crystal families of space E^n

In paper XI (Veysseyre, Weigel & Phan, 1993), we gave a definition of the gZ-red. and of the gZ-irr. crystal families connected to the bases of the irreducible representations of their holohedries. We have explained the building of the primitive cell of a gZ-red. family as the rectangular product of two or more subcells. In this section, we give another property of the gZ-red. crystal families. First, let us consider the translation group of the primitive lattice of the gZ-red. tetragonal family $u\mathbf{a}_1+v\mathbf{a}_2+w\mathbf{a}_3$, where $(u, v, w)\in Z^3$: $\mathbf{a}_1\perp\mathbf{a}_2$ and $\|\mathbf{a}_1\|=\|\mathbf{a}_2\|$; $\mathbf{a}_3\perp\mathbf{a}_1$ and $\mathbf{a}_3\perp\mathbf{a}_2$. It can be considered as the direct sum $\oplus$ of two translation groups of two orthogonal primitive lattices: that of the square family $u\mathbf{a}_1+v\mathbf{a}_2$ and that of the segment family of E^1, $w\mathbf{a}_3$. Therefore, we can write

$$u\mathbf{a}_1\oplus v\mathbf{a}_2\oplus w\mathbf{a}_3=(u\mathbf{a}_1+v\mathbf{a}_2)\oplus w\mathbf{a}_3.$$

Note that the reducibility of the tetragonal family into gZ-irr. subfamilies (square family of E^2 and segment family of E^1) is possible in only one way.

In the same way, the only possible reduction of the gZ-red. family number XV of E^4, *viz* the hexagon square family, into gZ-irr. subfamilies is

$$u\mathbf{a}_1\oplus v\mathbf{a}_2\oplus w\mathbf{a}_3\oplus s\mathbf{a}_4=(u\mathbf{a}_1\oplus u\mathbf{a}_2)\oplus(w\mathbf{a}_3\oplus s\mathbf{a}_4)$$

where

$$(u, v)\in Z^2, (w, s)\in Z^2$$

and

$$\|\mathbf{a}_1\|=\|\mathbf{a}_2\|, \|\mathbf{a}_3\|=\|\mathbf{a}_4\|.$$

$u\mathbf{a}_1\oplus v\mathbf{a}_2$ is the translation group of the hexagon family of E^2 and $w\mathbf{a}_3\oplus s\mathbf{a}_4$ that of the square family of E^2 (Fig. 1). We say that the gZ-red. family hexagon square of E^4 (family number XV) is the orthogonal product of two gZ-irr. subfamilies of E^2, the hexagon square of E^4 [hexagon (of E^2)$\oplus$square (of E^2)],

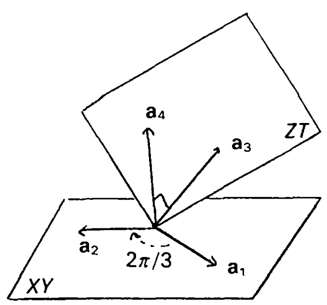

Fig. 1. Cell of the lattice of the hexagon square family of E^4. The two planes (XY) and (ZT) are orthogonal; they intersect only at one point. $\|\mathbf{a}_1\|=\|\mathbf{a}_2\|$, $\|\mathbf{a}_3\|=\|\mathbf{a}_4\|$, where $\|\ \|$ means Euclidean norm.

according to the formula

$$F^4_{\mathrm{XV}} = F^2_{\mathrm{III}} + F^2_{\mathrm{IV}}.$$

F^n_p means the family number p of space E^n, of dimension n, according to the nomenclature of Brown, Bülow Neubüser, Wondratschek & Zassenhaus (1978) (see Tables 1 and 2).

As a counter-example, let us consider the orthogonal di iso square family, *i.e.* the hypercubic 4 family of E^4. There is no unique way to reduce it into two orthogonal subfamilies; there are three possible ways to reduce it into two isosquare subfamilies [XY and ZT, XZ and YT, XT and YZ, if (X, Y, Z, T) are the vectors that define the primitive cell of this family]. This family is gZ-irr. of type 4 (Veysseyre *et al.* 1993).

Lastly, let us remember another fundamental property of the holohedry of a gZ-red. family: it is the direct product of the holohedries of the gZ-irr. subfamilies that appear in the construction of the gZ-red. family. This property is obvious in the WPV symbols. For instance, $4mm$ is the symbol of the point-symmetry group of the square, $6mm$ is that of the hexagon and m is that of the segment, therefore $4mm \perp m$ is the WPV symbol of the holohedry of the tetragonal family, $6mm \perp 4mm$ that of the hexagon square family of E^4.

Now, we can generalize these properties for any gZ-red. crystal family F^n_p of E^n. Indeed, each gZ-red. crystal family of space E^n can be split in a unique way, except for the order, as follows:

$$F^n_p = F^i_q \oplus F^j_r \oplus F^k_s \oplus \ldots, \qquad (1)$$

where $n, i, j, p, q, r, \ldots$ are integers, $i \geq j \geq k \geq \ldots$. $E^n = E^i \oplus E^j \oplus E^k \ldots$ means that the space E^n is the direct sum of subspaces $E^i, E^j, \ldots$ of dimensions $i, j, \ldots$, two-by-two orthogonal. F^i_q, F^j_r, F^h_s are gZ-irr. families belonging to spaces E^i, E^j, E^k, respectively. F^i_q is the gZ-irr. family of space E^i numbered q, and so on ....

As previously, the direct product of the different lattice-translation groups of the gZ-irr. subfamilies F^i_q, $F^j_r, \ldots$ is identical with the lattice translation group of the gZ-red. family F^n_p. The holohedries,* *i.e.* the point groups of the empty lattices, have the same property.

Now, we establish precise rules for giving a name to the gZ-red. families. We have already noticed that the order of the terms in (1) is of no importance.

Rule 1. The name of the gZ-red. crystal family F^n_p is the succession of the names of the gZ-irr. crystal subfamilies F^i_q, $F^j_r, \ldots$, the adjective 'orthogonal' being omitted.

Let us give an example. Consider $F^5_{\mathrm{XVIII}} = F^3_{\mathrm{VI}} + F^2_{\mathrm{IV}}$ (and $E^5 = E^3 \oplus E^2$), where F^3_{VI} is the cubic family of E^3 and F^2_{IV} is the hexagon family of E^2. The name of the family F^5_{XVIII} is thus 'cubic hexagon', which means orthogonal cubic hexagon. However, this decomposition is commutative; the name of this family could alternatively be 'hexagon cubic'. Therefore, we suggest the adoption of the following arbitrary order for the name of a gZ-red. crystal family.

Rule 2. If all the indices $i, j, k, \ldots$ in (1) are different, there is no difficulty or ambiguity because we supposed $i \geq j \geq k \ldots$ (hence $i > j > k \ldots$)

Therefore, the name for the family F^5_{XVIII} is cubic hexagon.

Now, let us suppose that two or more indices are equal. Consider $F^n_p = F^i_q \oplus F^i_r \oplus F^i_s \oplus \ldots$.

(*a*) If q, r and s are different, we start with the name of the subfamily whose holohedry is of highest order. For this reason, the name of the family F^4_{XV} is hexagon square and not square hexagon because the order of the holohedry 'hexagon' is 12 and that of the square is 8 (Tables 1 and 2).

(*b*) If $q, r, s, \ldots$ are equal, we shorten the names with the prefix di or tri ...; for instance, di cubic family or tri hexagon family, not cubic cubic family *etc.*

Rule 3. There exists only one crystal family in space E^1 (of dimension 1), *viz* F^1_{I}, which is obviously irreducible, of type 1. We explain our choices through some examples.

(*a*) $F^n_p = F^i_p \oplus F^j_r \oplus \ldots \oplus F^1_{\mathrm{I}}$,

$$E^n = E^i \oplus E^j \oplus \ldots \oplus E^1.$$

Only one space, E^1, occurs in the decomposition. Then the name of the crystal family F_p is

(name of the gZ-red. crystal family of space E^{n-1})-al,

'al' being the abbreviation of orthogonal. The cell built in this way is a right hyperprism. Three particular examples are given.

(1) $F^3_{\mathrm{V}} = F^2_{\mathrm{IV}} \oplus F^1_{\mathrm{I}}$,
F^3_{V} is the hexagonal family; its cell is a right prism based on (a third of) a hexagon;

(2) $F^4_{\mathrm{XVII}} = F^3_{\mathrm{VI}} \oplus F^1_{\mathrm{I}}$,
F^4_{XVII} is the cubic-al family (Tables 1 and 2); its cell is a right hyperprism based on a cube;

(3) $F^5_{\mathrm{XX}} = F^4_{\mathrm{XV}} \oplus F^1_{\mathrm{I}}$,
F^5_{XX} is the hexagon square-al family; its cell is a right hyperprism based on a hexagon square.

* Generally, the holohedry of a crystal family is the point group of the primitive empty lattice. There are a few exceptions. For instance, the order of the holohedry of the hypercubic 4 centred lattice is $1152 = 3 \times 384$, whereas the order of the holohedry of the primitive hypercubic 4 lattice is 384. We explain this property (Veysseyre, Weigel, Phan & Effantin, 1984) by the Pythagorus theorem: the half hyperdiagonal of the hypercube is equal to its edge length if and only if, for $n = 4$, $(a^2/4 + a^2/4 + a^2/4 + a^2/4)^{1/2} = a$.

(*b*) $F_p^n = F_q^i \oplus F_r^j \oplus \ldots \oplus F_{\mathrm{I}}^1 \oplus F_{\mathrm{I}}^1$.
Then, the name of the family F_p^n is the name of the subfamily of space E^{n-2} followed by 'rectangle', for example, the family F_{XLVIII}^6 of space E^6, a hexagon square rectangle family, corresponds to the splitting

$$F_{\mathrm{XLVIII}}^6 = F_{\mathrm{XV}}^4 \oplus F_{\mathrm{I}}^1 \oplus F_{\mathrm{I}}^1.$$

All crystal families of space E^6 will be listed in a forthcoming paper.

(*c*) $F_p^n = F_q^i \oplus F_r^j \oplus \ldots \oplus F_{\mathrm{I}}^1 \oplus F_{\mathrm{I}}^1 \oplus F_{\mathrm{I}}^1$, where $i > j > \ldots > 1$.
Then, the name of the family F_p^n is the name of the subfamily of space E^{n-3} followed by 'orthorhombic'. As an example, we give the F_{LIII}^6 cubic orthorhombic family (see table 1 and paper XIII):

$$F_{\mathrm{LIII}}^6 = F_{\mathrm{VI}}^3 \oplus F_{\mathrm{I}}^1 \oplus F_{\mathrm{I}}^1 \oplus F_{\mathrm{I}}^1.$$

(*d*) Lastly, if the number of subfamilies of type F_{I}^1 is higher than three, the name of the family F_p^n ends with 'orthotopic *h*', where *h* is the number of F_{I} occurring in the general formula. We take a family of space E^7 to illustrate this case:

$$F_p^7 = F_{\mathrm{I}}^3 \oplus F_{\mathrm{I}}^1 \oplus F_{\mathrm{I}}^1 \oplus F_{\mathrm{I}}^1 \oplus F_{\mathrm{I}}^1$$

is the triclinic orthotopic 4 family.

III. Counting and names of the gZ-irreducible crystal families

In this section, we explain how gZ-irr. crystal families are generated either by the transitive crystal rotations or by the product of transitive crystal rotations of the same angle.

First, we recall some properties of transitive crystal rotations.

III.1. *The crystal (proper) rotations*

According to Hermann terminology (Hermann, 1949), a transitive crystal PSO has only one class of conjugate roots. A transitive crystal PSO of finite order *m* only occurs in a space of dimension *n* equal to $\varphi(m)$, where $\varphi(m)$ is the Euler indicatrix of *m*, *i.e.* the number of integers prime with *m* and less than *n*. Bertaut (1988) established two scaling laws, which are

$$\varphi(2m) = \varphi(m) \quad \text{for } m \text{ odd},$$

$$\varphi(2m) = 2\varphi(m) \quad \text{for } m \text{ even},$$

e.g.

$$1 = \varphi(1) = \varphi(2),$$

$$2 = \varphi(3) = \varphi(6) = \varphi(4),$$

$$4 = \varphi(5) = \varphi(10) = \varphi(8) = \varphi(12),$$

$$6 = \varphi(7) = \varphi(14) = \varphi(9) = \varphi(18).$$

Table 4. *Number and symbols of transitive and intransitive crystal rotations*

	Transitive rotations	Intransitive rotations	Number
Space E^2	4; 3; 6	1; 2	5
Space E^3		1; 2; 3; 4; 6	5
Space E^4	88; 55; 10_2(1010) 12_2(1212)	1; 2; 3; 4; 6 $\bar{1}_4$ = 22; 23; 24; 26; 33; 34; 36; 44; 46; 66	19
Space E^5		The 19 crystal rotations of space E^4	19

Since $\varphi(3) = 2$, a threefold rotation is a transitive crystal PSO in space E^2, whereas it is an 'intransitive' crystal PSO in space E^3 (Hermann terminology). More accurately, in a space of dimension *n*, a crystal PSO that has only one class of conjugate roots may be a transitive crystal PSO if its order satisfies $\varphi(m) = n$ or an intransitive PSO if its order satisfies $\varphi(m) < n$. A crystal PSO that has several classes of conjugate roots or has multiple roots is an intransitive PSO. For instance, in space E^4, the double rotations 24, $22 = \bar{1}_4$, 34, ... are intransitive rotations, whereas the double rotations 55, 88, 10_2(10 10) and 12_2(12 12) are transitive rotations. All these double rotations appear for the first time in space E^4. Let us note that the simple crystal rotations 5, 8, 10, 12 are forbidden in E^n, whatever *n* (for the crystals but not for the quasi-crystals).

The transitive and intransitive crystal rotations of spaces E^2, E^3, E^4 and E^5 are listed in Table 4. Fig. 2 illustrates the double rotation 6^16^1 of order six $[6^16^1]^6 = 1$ (identity).

A significant property of a transitive crystal rotation of space E^n can be pointed out: it does not leave invariant any subspace of dimension lower than *n* except the point chosen as origin. For instance, 4_{xy}^1 is transitive in space E_{xy} but intransitive in space E_{xyz} (the axis *z* is invariant) and in space E_{xyzt} [the plane (*zt*) is invariant]. This property explains why the gZ-irr. crystal families are generated by the transitive

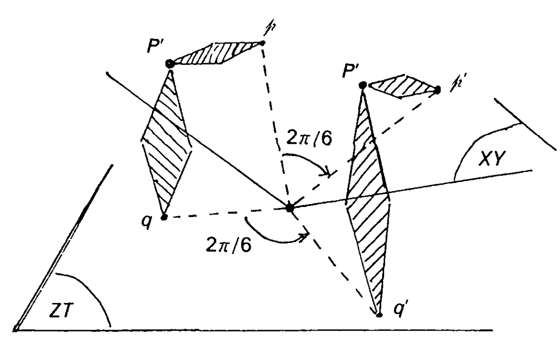

Fig. 2. Double crystallographic rotation 6^16^1. p and q are the projections of P onto the planes XY and ZT, which are orthogonal. Rotation by $2\pi/6$ into the plane XY: $p \to p'$. Rotation by $2\pi/6$ into the plane ZT: $q \to q'$. P' is the point for which the projections are p' and q'. Double rotation 6^16^1: $P \to P'$.

crystal rotations. Another type of crystal rotation generates gZ-irr. crystal families; they are the rotations called 'degenerate' in a previous paper (Weigel, Veysseyre, Phan, Effantin & Billiet, 1984). Degenerate means that a root of the characteristic polynomial is of multiple order. Examples in space E^4 are $\bar{1}_4=(22)$, 33*, 44*, 66*. We now describe a method for constructing gZ-irr. crystal families of space E^4.

III.2. *From the transitive crystal rotations and the degenerate crystal rotations to the gZ-irr. crystal families*

We explain this method through three examples, described below. The general form of the metric tensor of a crystal cell $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ of space E^4 is (see footnote † on p. 481 of paper XI)

$$\begin{pmatrix} a & A & B & C \\ A & b & D & E \\ B & D & c & F \\ C & E & F & d \end{pmatrix}.$$

The lower-case letters denote the squared norms of the sides:

$$\|\mathbf{a}_1\|^2=a, \quad \|\mathbf{a}_2\|^2=b, \quad \|\mathbf{a}_3\|^2=c, \quad \|\mathbf{a}_4\|^2=d.$$

The capital letters denote all the scalar products of two vectors $\mathbf{a}_i$, $\mathbf{a}_j$: $\mathbf{a}_1\cdot\mathbf{a}_2=A, \ldots$.

(1) We start with the group generated by the degenerate rotation $4_{xy}4_{zt}$, *viz* a group of order 4 (4^14^1; $\bar{1}_4$; $4^{-1}4^{-1}$; 1). The matrix of the PSO 4^14^1 is

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We suppose that the two planes $(\mathbf{a}_1\mathbf{a}_2)$ and $(\mathbf{a}_3\mathbf{a}_4)$ are not orthogonal. Any point of the lattice defined by vectors $\mathbf{a}_i$ can be written as

$$\overrightarrow{OM}=\sum_i n_i\mathbf{a}_i \quad \text{where } n_i\in Z.$$

If f denotes the PSO $4^1_{xy}4^1_{zt}$, we can write

$$f(\overrightarrow{OM})=-n_2\mathbf{a}_1+n_1\mathbf{a}_2-n_4\mathbf{a}_3+n_3\mathbf{a}_4,$$

f being an isometry:

$$\|\overrightarrow{OM}\|^2=\|f(\overrightarrow{OM})\|^2 \quad \forall n_i\in Z.$$

Easy calculations lead to the results

$$\begin{aligned}\|\overrightarrow{OM}\|^2 &= an_1+bn_2+cn_3+dn_4+2n_1n_2A \\ &\quad +2n_1n_3B+2n_1n_4C+2n_2n_3D \\ &\quad +2n_2n_4E+2n_3n_4F\end{aligned}$$

and

$$\begin{aligned}\|f(\overrightarrow{OM})\|^2 &= an_2+bn_2+cn_4+dn_3-2n_1n_2A \\ &\quad +2n_2n_4B-2n_2n_3C-2n_1n_4D \\ &\quad +2n_1n_3E-2n_3n_4F.\end{aligned}$$

Hence,

$$a=b, \qquad c=d,$$
$$A=F=0, \qquad C=-D, \qquad B=E.$$

The metric tensor in this case is

$$\begin{pmatrix} a & 0 & B & C \\ 0 & a & -C & B \\ B & -C & c & 0 \\ C & B & 0 & c \end{pmatrix}$$

It is the metric tensor of a crystal family called 'diclinic di square'.

We can add two rotations, for instance a twofold rotation 2_{yt} denoted g:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We generate a group of order 8,

$$g(\overrightarrow{OM})=n_1\mathbf{a}_1-n_2\mathbf{a}_2+n_3\mathbf{a}_3-n_4\mathbf{a}_4.$$

A similar calculation ($\|g(\overrightarrow{OM})\|^2=\|\overrightarrow{OM}^2\|$) gives the results

$$a=b, \qquad c=d,$$
$$A=F=C=D=0, \qquad B=E.$$

The metric tensor is then

$$\begin{pmatrix} a & 0 & B & 0 \\ 0 & a & 0 & B \\ B & 0 & c & 0 \\ 0 & B & 0 & c \end{pmatrix}.$$

It is the metric tensor of a crystal family called 'monoclinic di square'. The type of the gZ-irr. of these two families has been studied.

The rotations 33* and 66* can be similarly studied and lead to the crystal families diclinic di hexagon and monoclinic di hexagon.

(2) We now consider the transitive crystal rotation 88. With respect to the general basis $(\mathbf{a}_i)$, the matrix of this isometry, denoted f, is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Table 5. *Geometrically Z-irreducible crystal families of E^4*

The double rotations through the same angle generate gZ-irr. crystal families. The transitive double rotations 88, 55, 10_2 and 12_2 appear in the second part of the table.

Double crystal rotations	gZ-irr. crystal families	Type	Symbol
$\bar{1}_4(22)$	Hexaclinic	1	$\overline{1,1,1,1}$
44	Diclinic di square	4	$\overline{2,2'}$
	Monoclinic di square	3	$\overline{2,2}$
33 and 66	Diclinic di hexagon	4	$\overline{2,2'}$
	Monoclinic di hexagon	3	$\overline{2,2}$
88 (and 44)	Monoclinic di iso square (octodic)	5	4′
	Hypercubic 4	2	4
12_2 (and 66 and 33)	Monoclinic di iso hexagon (dodecadic)	5	4′
	Di iso hexagon	2	4
55 and 10_2	Decadic	5	4′
	Rhombotopic $-\frac{1}{4}$	2	4

Similar calculations give, with respect to the basis $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$, the metric tensor

$$\begin{pmatrix} a & A & 0 & -A \\ A & a & A & 0 \\ 0 & A & a & A \\ -A & 0 & A & a \end{pmatrix}$$

or, with respect to the basis $(\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_2, \mathbf{a}_4)$,

$$\begin{pmatrix} a & 0 & A & -A \\ 0 & a & A & A \\ A & A & a & 0 \\ -A & A & 0 & a \end{pmatrix}.$$

This is the metric tensor of the cell of the 'monoclinic di iso square' family. Obviously, if $A=0$, we obtain the 'hypercubic family'.

(3) The third example involves the transitive crystal rotations 55 and 10_2. Similar calculations to the previous ones with the double-rotation $5^1 5^3$ matrix

$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

lead to the metric tensor

$$\begin{pmatrix} a & A & -\frac{1}{2}(a+2A) & -\frac{1}{2}(a+2A) \\ A & a & A & -\frac{1}{2}(a+2A) \\ -\frac{1}{2}(a+2A) & A & a & A \\ -\frac{1}{2}(a+2A) & -\frac{1}{2}(a+2A) & A & a \end{pmatrix}.$$

This is the metric tensor of the crystal family called 'decadic' (Table 2). A particular case is the 'rhombotopic $-\frac{1}{4}$' family if $A=-\frac{1}{2}(a+2A)$ or $A=-a/4$.

Table 6. *Geometrically Z-reducible crystal families generated by double rotations in E^4*

Double crystal rotations	Names of the gZ-red. crystal families	WPV symbols of the holohedries
42	Square oblic	$4mm \perp 2$
62 and 32	Hexagon oblic	$6mm \perp 2$
64 and 43	Hexagon square	$6mm \perp 4mm$
63	Di hexagon	$63, 2, 2.\bar{1}_4$

All these results are summarized in Table 5. The 11 gZ-irr. crystal families of space E^4 are generated by the double rotations through the same angle; the four transitive crystal rotations are among them. We note that the gZ-irr. crystal families generated by the transitive rotations (lower part of Table 5) belong to the types 2 and 5 of irreducibility denoted 4 and 4′ in space E^4, the other ones belong to the types 1, 3 and 4, denoted respectively $\overline{1,1,1,1}$, $\overline{2,2}$ and $\overline{2,2'}$ (Veysseyre *et al.*, 1993). The transitivity of the PSOs leads to an irreducibility symbol with one number only. As a counter-example, we list the gZ-red. crystal families generated by the double rotations through different angles (Table 6); the name of the family recalls the double rotation as far as possible. The 14 types of crystal double rotations are listed in Tables 5 and 6.

Concluding remarks

In this paper, we have listed the crystal families of spaces E^2, E^3, E^4, E^5; we kept the order given by Brown *et al.* (1978) for space E^4 and by Plesken (1981) for space E^5 and by Plesken & Hanrath (1984) for space E^6. Nevertheless, we gave names and symbols to holohedries connected to our geometrical method (splitting of the space, type of the crystal rotations that generate the families). Similar results for spaces E^6 and E^7 will be published in papers XIII and XIV.

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